Extending the Cooperative Dual-Task Space in Conformal Geometric Algebra

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Appendix

A. Derivation of the Jacobian of the exponential map

The exponential map is a mapping from bivectors to motors, i.e. $B = \log(M)$. The parameters of the motor and bivector are as follows

$$M = m_1 + m_2\boldsymbol{e}_{23} + m_3\boldsymbol{e}_{13} + m_4\boldsymbol{e}_{12} + m_5\boldsymbol{e}_{1\infty} + m_6\boldsymbol{e}_{2\infty} + m_7\boldsymbol{e}_{3\infty} + m_8\boldsymbol{e}_{123\infty},$$

and

$$B = b_1\boldsymbol{e}_{23} + b_2\boldsymbol{e}_{13} + b_3\boldsymbol{e}_{12} + b_4\boldsymbol{e}_{1\infty} + b_5\boldsymbol{e}_{2\infty} + b_6\boldsymbol{e}_{3\infty}.$$

Since a motor is a product of a translator T and a rotor R, we can find the exponential map

$$M = TR,$$

$$T = \left(1 - \frac{1}{2}(b_4\boldsymbol{e}_{1\infty} + b_5\boldsymbol{e}_{2\infty} + b_6\boldsymbol{e}_{3\infty})\right),$$

$$R = \frac{1}{\theta}\left(\cos(\frac{1}{2}\theta) - \sin(\frac{1}{2}\theta)\,(b_1\boldsymbol{e}_{23} + b_2\boldsymbol{e}_{13} + b_3\boldsymbol{e}_{12})\right),$$

where

$$\theta = \sqrt{b_1^2 + b_2^2 + b_3^2}.$$

In terms of parameters the m_i and b_i this becomes

$$\begin{aligned}
m_1 &= \frac{1}{\sqrt{b_1^2+b_2^2+b_3^2}}\cos\left(\frac{1}{2}\sqrt{b_1^2+b_2^2+b_3^2}\right),\\
m_2 &= \frac{-b_1}{b_1^2+b_2^2+b_3^2}\sin\left(\frac{1}{2}\sqrt{b_1^2+b_2^2+b_3^2}\right),\\
m_3 &= \frac{-b_2}{b_1^2+b_2^2+b_3^2}\sin\left(\frac{1}{2}\sqrt{b_1^2+b_2^2+b_3^2}\right),\\
m_4 &= \frac{-b_3}{b_1^2+b_2^2+b_3^2}\sin\left(\frac{1}{2}\sqrt{b_1^2+b_2^2+b_3^2}\right),\\
m_5 &= \frac{1}{2}\left(m_1b_4 + m_3b_6 + m_4b_5\right),\\
m_6 &= \frac{1}{2}\left(m_1b_5 + m_2b_6 - m_4b_4\right),\\
m_7 &= \frac{1}{2}\left(m_1b_5 - m_2b_5 - m_3b_4\right),\\
m_8 &= \frac{1}{2}\left(m_2b_4 - m_3b_5 + m_4b_6\right).
\end{aligned}$$

The non-trivial partial derivatives can then be found as

$$\begin{aligned}
\frac{\partial m_1}{\partial b_1} &= -\frac{1}{2}\frac{b_1}{\theta}\sin\left(\frac{1}{2}\theta\right),\\
\frac{\partial m_1}{\partial b_2} &= -\frac{1}{2}\frac{b_2}{\theta}\sin\left(\frac{1}{2}\theta\right),\\
\frac{\partial m_1}{\partial b_3} &= -\frac{1}{2}\frac{b_3}{\theta}\sin\left(\frac{1}{2}\theta\right),\\
\frac{\partial m_2}{\partial b_1} &= \sin\left(\frac{1}{2}\theta\right)\left(\frac{b_1^2}{\theta^3} - \frac{1}{\theta}\right) - \frac{1}{2}\frac{b_1^2}{\theta^2}\cos\left(\frac{1}{2}\theta\right),\\
\frac{\partial m_2}{\partial b_2} &= b_1\left(\frac{b_2}{\theta^3}\sin\left(\frac{1}{2}\theta\right) - \frac{1}{2}\frac{b_2}{\theta^2}\cos\left(\frac{1}{2}\theta\right)\right),\\
\frac{\partial m_2}{\partial b_3} &= b_1\left(\frac{b_3}{\theta^3}\sin\left(\frac{1}{2}\theta\right) - \frac{1}{2}\frac{b_3}{\theta^2}\cos\left(\frac{1}{2}\theta\right)\right),\\
\frac{\partial m_3}{\partial b_1} &= b_2\left(\frac{b_1}{\theta^3}\sin\left(\frac{1}{2}\theta\right) - \frac{1}{2}\frac{b_1}{\theta^2}\cos\left(\frac{1}{2}\theta\right)\right),\\
\frac{\partial m_3}{\partial b_2} &= \sin\left(\frac{1}{2}\theta\right)\left(\frac{b_2^2}{\theta^3} - \frac{1}{\theta}\right) - \frac{1}{2}\frac{b_2^2}{\theta^2}\cos\left(\frac{1}{2}\theta\right),\\
\frac{\partial m_3}{\partial b_3} &= b_2\left(\frac{b_3}{\theta^3}\sin\left(\frac{1}{2}\theta\right) - \frac{1}{2}\frac{b_3}{\theta^2}\cos\left(\frac{1}{2}\theta\right)\right),\\
\frac{\partial m_4}{\partial b_1} &= b_3\left(\frac{b_1}{\theta^3}\sin\left(\frac{1}{2}\theta\right) - \frac{1}{2}\frac{b_1}{\theta^2}\cos\left(\frac{1}{2}\theta\right)\right),\\
\frac{\partial m_4}{\partial b_2} &= b_3\left(\frac{b_2}{\theta^3}\sin\left(\frac{1}{2}\theta\right) - \frac{1}{2}\frac{b_2}{\theta^2}\cos\left(\frac{1}{2}\theta\right)\right),\\
\frac{\partial m_4}{\partial b_3} &= \sin\left(\frac{1}{2}\theta\right)\left(\frac{b_3^2}{\theta^3} - \frac{1}{\theta}\right) - \frac{1}{2}\frac{b_3^2}{\theta^2}\cos\left(\frac{1}{2}\theta\right),\\
\frac{\partial m_5}{\partial b_1} &= -\frac{1}{2}\left(b_4\frac{\partial m_1}{\partial b_1} - b_6\frac{\partial m_3}{\partial b_1} - b_5\frac{\partial m_4}{\partial b_1}\right),\\
\frac{\partial m_5}{\partial b_2} &= -\frac{1}{2}\left(b_4\frac{\partial m_1}{\partial b_2} - b_6\frac{\partial m_3}{\partial b_2} - b_5\frac{\partial m_4}{\partial b_2}\right),\\
\frac{\partial m_5}{\partial b_3} &= -\frac{1}{2}\left(b_4\frac{\partial m_1}{\partial b_3} - b_6\frac{\partial m_3}{\partial b_3} - b_5\frac{\partial m_4}{\partial b_3}\right),\\
\frac{\partial m_6}{\partial b_1} &= -\frac{1}{2}\left(b_5\frac{\partial m_1}{\partial b_1} - b_6\frac{\partial m_2}{\partial b_1} + b_4\frac{\partial m_4}{\partial b_1}\right),\\
\frac{\partial m_6}{\partial b_2} &= -\frac{1}{2}\left(b_5\frac{\partial m_1}{\partial b_2} - b_6\frac{\partial m_2}{\partial b_2} + b_4\frac{\partial m_4}{\partial b_2}\right),\\
\frac{\partial m_6}{\partial b_3} &= -\frac{1}{2}\left(b_5\frac{\partial m_1}{\partial b_3} - b_6\frac{\partial m_2}{\partial b_3} + b_4\frac{\partial m_4}{\partial b_3}\right),\\
\frac{\partial m_7}{\partial b_1} &= -\frac{1}{2}\left(b_6\frac{\partial m_1}{\partial b_1} + b_5\frac{\partial m_2}{\partial b_1} + b_4\frac{\partial m_3}{\partial b_1}\right),\\
\frac{\partial m_7}{\partial b_2} &= -\frac{1}{2}\left(b_6\frac{\partial m_1}{\partial b_2} + b_5\frac{\partial m_2}{\partial b_2} + b_4\frac{\partial m_3}{\partial b_2}\right),\\
\frac{\partial m_7}{\partial b_3} &= -\frac{1}{2}\left(b_6\frac{\partial m_1}{\partial b_3} + b_5\frac{\partial m_2}{\partial b_3} + b_4\frac{\partial m_3}{\partial b_3}\right),\\
\frac{\partial m_8}{\partial b_1} &= -\frac{1}{2}\left(b_4\frac{\partial m_2}{\partial b_1} - b_5\frac{\partial m_3}{\partial b_1} + b_6\frac{\partial m_4}{\partial b_1}\right),
\end{aligned}$$

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\begin{aligned}
\frac{\partial m_8}{\partial b_2} &= -\frac{1}{2}\left(b_4\frac{\partial m_2}{\partial b_2} - b_5\frac{\partial m_3}{\partial b_2} + b_6\frac{\partial m_4}{\partial b_2}\right), \\
\frac{\partial m_8}{\partial b_3} &= -\frac{1}{2}\left(b_4\frac{\partial m_2}{\partial b_3} - b_5\frac{\partial m_3}{\partial b_3} + b_6\frac{\partial m_4}{\partial b_3}\right), \\
\frac{\partial m_5}{\partial b_4} &= -\frac{1}{2}m_1, \\
\frac{\partial m_5}{\partial b_5} &= \frac{1}{2}m_4, \\
\frac{\partial m_5}{\partial b_6} &= \frac{1}{2}m_3, \\
\frac{\partial m_6}{\partial b_4} &= -\frac{1}{2}m_4, \\
\frac{\partial m_6}{\partial b_5} &= -\frac{1}{2}m_1, \\
\frac{\partial m_6}{\partial b_6} &= \frac{1}{2}m_2, \\
\frac{\partial m_7}{\partial b_4} &= -\frac{1}{2}m_3, \\
\frac{\partial m_7}{\partial b_5} &= -\frac{1}{2}m_2, \\
\frac{\partial m_7}{\partial b_6} &= -\frac{1}{2}m_1, \\
\frac{\partial m_8}{\partial b_4} &= -\frac{1}{2}m_2, \\
\frac{\partial m_8}{\partial b_5} &= \frac{1}{2}m_3, \\
\frac{\partial m_8}{\partial b_6} &= -\frac{1}{2}m_4.
\end{aligned}
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